COMBINATORICA 5 (4) (1985) 283—293

SOME REMARKS ON UNIVERSAL GRAPHS

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Received 12 January 1984
Revised 1 February 1985

Dedicated to Klaus Wagner on his 75th birthday

Let Γ be a class of countable graphs, and let $\mathcal{G}(\Gamma)$ denote the class of all countable graphs that do not contain any subgraph isomorphic to a member of Γ. Furthermore, let $T\Gamma$ and $H\Gamma$ denote the class of all subdivisions of graphs in Γ and the class of all graphs contracting to a member of Γ, respectively. As the main result of this paper it is decided which of the classes $\mathcal{G}(TK^n)$ and $\mathcal{G}(HK^n)$, $n\leqq\aleph_0$, contain a universal element. In fact, for $\mathcal{G}(TK^4)=\mathcal{G}(HK^4)$ a strongly universal graph is constructed, whereas for $5\leqq n\leqq\aleph_0$ the classes $\mathcal{G}(TK^n)$ and $\mathcal{G}(HK^n)$ have no universal elements.

1. Introduction

A member G^* of a class $\mathcal{G}$ of graphs is called (strongly) *universal* in $\mathcal{G}$ if every $G\in\mathcal{G}$ is isomorphic to some (induced) subgraph of G^*; the mapping $V(G)\to V(G^*)$ induced by such an isomorphism is called a (strong) embedding of G in G^*.

The concept of a universal graph was first introduced by R. Rado [10], who constructed a graph R strongly universal in the class $\mathcal{G}_0$ of all countable graphs, The main motivation to look for a (strongly) universal graph G^* in a class $\mathcal{G}$ of graphs is that we may get information from it on the general structure of the elements of $\mathcal{G}$, since these all appear as certain (induced) subgraphs of G^*.

All graphs we consider in our investigations are countable, so we make the convention that throughout this paper 'graph' means 'finite or countably infinite graph'.

Let us call a class $\mathcal{G}$ of graphs monotone if $G'\subset G\in\mathcal{G}$ implies $G'\in\mathcal{G}$. If G^* is (strongly) universal in a monotone class $\mathcal{G}$, we get precisely the elements of $\mathcal{G}$, up to isomorphism, as the (induced) subgraphs of G^*. Thus by knowing G^* we get a survey of $\mathcal{G}$.

On the other hand, for a number of non-monotone classes of graphs one easily finds a strongly universal graph G^* simply by extending the Rado graph R in a suitable way and thereby turning it into a member of $\mathcal{G}$. For if $R\subset G^*\in\mathcal{G}$, then G^*, too, contains all countable graphs, in particular those of $\mathcal{G}$. In this manner strongly universal elements can easily be obtained for the class of all graphs with some given connectivity, say, or that of all graphs with a given finite group of automorphisms. However, it is clear that these universal graphs carry little or no information about the structure of the other elements of their respective classes.

AMS subject classification (1980): 05 C 75

It seems therefore appropriate to focus one's attention on monotone classes of graphs. These coincide with the classes $\mathcal{G}(\Gamma)$ mentioned in the abstract (i.e. the classes of graphs in which certain configurations are 'forbidden'); for each $\mathcal{G}(\Gamma)$ is obviously monotone, and a monotone class $\mathcal{H}$ is identical with $\mathcal{G}(\overline{\mathcal{H}})$, where $\overline{\mathcal{H}}$ is the class of all graphs not isomorphic to any member of $\mathcal{H}$.

In this paper the problem of (strongly) universal elements is solved for the classes $\mathcal{G}(TK^n)$ and $\mathcal{G}(HK^n)$ $(n \leqq \aleph_0)$, i.e. for the classes of graphs that do not contain a subdivision of, or subcontract to, a complete graph of order n for some given n.

Obviously the $\aleph_0$-regular tree is strongly universal in $\mathcal{G}(TK^3)=\mathcal{G}(HK^3)$. In Section 2 a strongly universal element of $\mathcal{G}(TK^4)=\mathcal{G}(HK^4)$ is constructed.

By means of simplicial decompositions and homomorphism (or subdivision) bases it is shown in Section 3 that for $5 \leqq n < \aleph_0$ no $\mathcal{G}(TK^n)$ or $\mathcal{G}(HK^n)$ contains a universal element. The theorem we obtain also covers a few other classes and includes Pach's result [9] that there is no universal planar graph.

In Section 4 we prove the following: whenever Γ is a non-empty class of graphs each containing an infinite path, the class $\mathcal{G}(\Gamma)$ has no universal element. From this follows in particular that $\mathcal{G}(TK^{\aleph_0})$ and $\mathcal{G}(HK^{\aleph_0})$ have no universal elements.

In the final section we discuss some of the implications of simplicial decompositions on the universal graph problem, and making use of the known homomorphism bases we obtain some further related results.

A number of open problems are also mentioned.

2. A Strongly Universal Graph in $\mathcal{G}(TK^4)$

Throughout this section we denote by $\mathcal{G}$ the class $\mathcal{G}(TK^4)=\mathcal{G}(HK^4)$ and by $\mathcal{G}^2$ the class of all 2-connected graphs in $\mathcal{G}$.

2.1. Definition. (i) Let G be a graph and $\mathcal{P}$ a set of finite paths in G. Call another set $L:=L(\mathcal{P})$ of finite paths in G a *labelling* of $\mathcal{P}$ if each path in L is non-trivial and contained in some path of $\mathcal{P}$; the elements of L are called labels of $\mathcal{P}$. A labelling L is *admissible* if $T \subset T'$ or $T' \subset T$ whenever $T, T' \in L$ are not edge-disjoint. If L is an (admissible) labelling of $\mathcal{P}$ we call $\mathcal{P}$ and each $P \in \mathcal{P}$ *(admissibly) labelled* by L.

(ii) Let H be a graph, $G \subset H$, and $\mathcal{P}$ an admissibly labelled set of finite paths in G. We call H an *admissible extension* of G with respect to $\mathcal{P}$ if there exists an admissibly labelled set $\mathcal{P}_H$ of independent $G-G$ paths in H of length at least 2 (i.e. paths whose endvertices are in G but whose inner vertices are not in G) such that

$$H = G \cup \bigcup_{P \in \mathcal{P}_H} P$$

and the endvertices of each $P \in \mathcal{P}_H$ coincide with the endvertices of some $T \in L(\mathcal{P})$.

(iii) An admissible extension as defined in (ii) is called *maximal* if the following holds for every $T \in L(\mathcal{P})$ with endvertices a, b:

Let $\tilde{P}$ be an arbitrary admissibly labelled path of length at least 2. Then $\mathcal{P}_H$ contains infinitely many paths P with endvertices a, b such that an isomorphism $\Phi\colon \tilde{P} \to P$ exists that maps the endvertices of $\tilde{P}$ onto a and b, and the labels of $\tilde{P}$ onto those of P.

The use of the term "maximal" in (2.1, iii) is justified by the following observation, which is immediate from the above definition.

2.2. If H and H^* are admissible extensions of G with respect to $\mathscr{P}$ and H^* is maximal, then there exists a strong embedding $\Phi\colon H \to H^*$ that fixes all vertices of G and respects the labelling of $\mathscr{P}_H$ (that is Φ maps the paths of $\mathscr{P}_H$ onto paths of $\mathscr{P}_{H^*}$ and the labels of each $P \in \mathscr{P}_H$ onto those of $\Phi(P)$). ∎

The following lemma states that all graphs constructed from an edge by repeated admissible extensions are in $\mathscr{G}^2$.

2.3. Lemma. *Let $G \in \mathscr{G}$ and $\mathscr{P}$ a set of paths in G such that either $G \simeq K^2$ with the admissible labelling $L(\mathscr{P}) = \mathscr{P} = \{G\}$ or $G' \subset G \in \mathscr{G}^2$ for some $G' \in \mathscr{G}$ where G is an admissible extension of G' and $\mathscr{P} = \mathscr{P}_G$. Then every admissible extension H of G with respect to $\mathscr{P}$, $|H| \geqq 3$, is contained in $\mathscr{G}^2$.*

Proof. Since any counterexample H is clearly 2-connected, it contains a TK^4; we can therefore choose H to have finite and minimal $\mathscr{P}_H$. For each $G-G$ path $P \in \mathscr{P}_H$ denote by $T(P)$ the element of $L(\mathscr{P})$ that forms a cycle together with P, and let $T_0 = T(P_0)$ be minimal among these $T(P)$'s with respect to inclusion. Then the inner vertices of T_0, as well as those of P_0, have degree 2 in H and can therefore not be branch vertices of a TK^4. Hence $H \setminus \mathring{P}_0$ contains a TK^4 (where $\mathring{P}_0$ denotes the interior of P_0), contradicting the choice of H. ∎

Let us now show the converse of Lemma 2.3, i.e. that every graph in $\mathscr{G}^2$ can be constructed by successive admissible extensions, starting from an edge.

2.4. Lemma. *Every $G \in \mathscr{G}^2$ can be expressed as $G = \bigcup_{i=1}^{\infty} G_i$ with $G_i \in \mathscr{G}^2$ for $i = 2, 3, \ldots$ in such a way that there exists a set $\mathscr{P}_0$ as well as sets $\mathscr{P}_i$ of independent $G_i - G_i$ paths in G for $i = 1, 2, \ldots$ such that*

(i) $G_1 \simeq K^2$,

(ii) $G_{i-1} = G_i \cup \bigcup_{P \in \mathscr{P}_i} P$, $i = 1, 2, \ldots$,

(iii) *G_{i+1} is an admissible extension of G_i with respect to $\mathscr{P}_{i-1}$, $i = 1, 2, \ldots$.*

Proof. Let $G_1 := G[x, y]$ for any edge xy of G and $\mathscr{P}_0 := \{G_1\}$. Having defined $G_1, \ldots, \ldots, G_i$ put

$$G_{i+1} := G_i \cup \bigcup_{P \in \mathscr{P}_i} P,$$

where $\mathscr{P}_i$ is any maximal set of independent chordless $G_i - G_i$ paths in G (a path is *chordless* if it contains every edge that joins two of its vertices other than its endvertices).

Clearly (i) and (ii) are satisfied for all i, so let us show (iii). For every $\mathscr{P}_{i-1}$, $i = 1, 2, \ldots$, define a labelling $L(\mathscr{P}_{i-1})$ as the set of all those subpaths T of some $P' \in \mathscr{P}_{i-1}$ that form a cycle together with some $P \in \mathscr{P}_i$. Since G_{i+1} contains no TK^4 (Fig. 1), $L(\mathscr{P}_{i-1})$ is admissible.

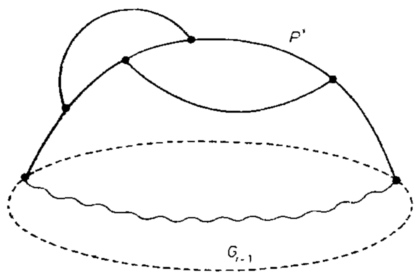

Fig. 1

To complete the proof of (iii) we yet have to show for $i \geqq 2$ that the endvertices of each $G_i - G_i$ path P in G lie indeed on a common path $P' \in \mathcal{P}_{i-1}$, which in turn implies $l(P) \geqq 2$, because P' was chordless and $\mathcal{P}_{i-1}$ maximal.

Now for every such P at least one of its endvertices a, b is contained in the interior $\mathring{P}'$ of some $P' \in \mathcal{P}_{i-1}$, because $\mathcal{P}_{i-1}$ was maximal; let a' and b' be the endvertices of P' and assume that $a \in \mathring{P}'$. Suppose $b \in G_i \setminus P'$. Since by our construction $G_i \setminus \mathring{P}'$ is 2-connected, $G_i \setminus \mathring{P}'$ contains independent paths $P_{a'}$ and $P_{b'}$ from b to a' and b', respectively. Moreover, there exists a (shortest) path P'' from $P_{a'} \setminus \{b\}$ to $P_{b'} \setminus \{b\}$ in $G_i \setminus (\mathring{P}' \cup \{b\})$, since otherwise b would separate a' and b'. The endvertices of P'' together with a and b are the branchvertices of a TK^4 in G (Fig. 2.). This completes the proof of (iii).

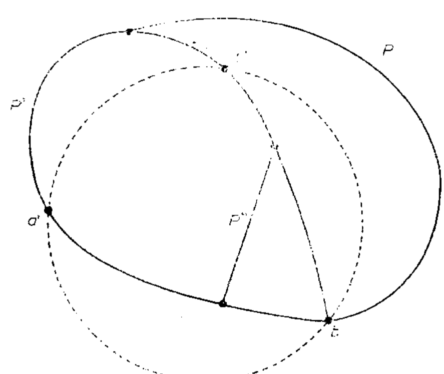

Fig. 2

All that remains to be shown is $G \subset \bigcup_{i=1}^{\infty} G_i$. Since G is 2-connected, we may for any given i and $e \in E(G)$ define $c_i(e)$ as the length of a shortest $G_i - G_i$ path containing e. In particular, we have $c_i(e) = 1$ for $e \in E(G_i)$.

Let $i \geqq 1$, $e \in E(G) \setminus E(G_i)$, and $c_i(e) =: l$. Let P be a $G_i - G_i$ path of length l containing e. Then e is contained in a unique $G_{i+1} - G_{i+1}$ path $P' \subset P$. By the maximality of $\mathcal{P}_i$, P is properly contained in P, implying $c_{i+1}(e) < l$.

Hence $c_i(e)$ decreases for each edge e of G strictly with increasing i until $e\in G_i$, so e is surely contained in $G_{c_1(e)}$. ▮

2.5. Theorem. $\mathcal{G}(TK^4)=\mathcal{G}(HK^4)$ *has a strongly universal element.*

Proof. Let us begin by defining a sequence $G_1^2\sqsubset G_2^2\sqsubset\dots$ of graphs, whose supremum $G^2:=\bigcup_{i=1}^{\infty} G_i^2$ is strongly universal in $\mathcal{G}^2$ ('$\sqsubset$' denotes 'induced subgraph').

Let $G_1^2\simeq K^2$ and put $L(\mathcal{P}_0^2)=\mathcal{P}_0^2=\{G_1^2\}$. $L(\mathcal{P}_0^2)$ is an admissible labelling of $\mathcal{P}_0^2$.

Having defined $G_1^2, \dots, G_i^2$ and $\mathcal{P}_0^2, \dots, \mathcal{P}_{i-1}^2$ as well as an admissible labelling of $\mathcal{P}_{i-1}^2$, we define G_{i+1}^2 as any maximal admissible extension of G_i^2 with respect to $\mathcal{P}_{i-1}^2$ and put $\mathcal{P}_i^2:=\mathcal{P}_{G_{i+1}^2}$. By Lemma 2.3 every G_i^2 is in $\mathcal{G}^2$ $(i\geqq 2)$, so also $G^2\in\mathcal{G}^2$.

To show the universality of G^2 in $\mathcal{G}^2$ let $G\in\mathcal{G}^2$ and $G=\bigcup_{i=1}^{\infty} G_i$ as in Lemma 2.4. We define a sequence $\Phi_1\subset\Phi_2\subset\dots$ of strong embeddings $\Phi_i\colon G_i\to G^2$ such that

(1) $\Phi_i(G_i)\subset G_i^2$, and Φ_i maps each path $P\in\mathcal{P}_{i-1}$ onto a path $P^2\in\mathcal{P}_{i-1}^2$ and the labels of every such P onto the labels of $\Phi_i(P)$ (i.e. for each $P\in\mathcal{P}_{i-1}$ and every $T\subset P$ we have $T\in L(\mathcal{P}_{i-1})$ iff $\Phi_i(T)\in L(\mathcal{P}_{i-1}^2)$).

Putting $\Phi_1(G_1)=G_1^2$, (1) is obviously satisfied for $i=1$. Suppose $\Phi_1, \dots, \Phi_i$ have been defined and (1) is satisfied for $1, \dots, i$. By Lemma 2.4, G_{i+1} is an admissible extension of G_i with respect to $\mathcal{P}_{i-1}$. Since Φ_i satisfies (1), G_{i+1} defines in a natural way an admissible extension $\tilde{G}_{i+1}^2$ of G_i^2 with respect to $\mathcal{P}_{i-1}^2$, which is by 2.2. strongly embedded in G_{i+1}^2. As this embedding can be chosen in such a way that it fixes G_i^2, $\tilde{G}_{i+1}^2$ induces an embedding $\Phi_{i+1}\colon G_{i+1}\to G_{i+1}^2$ that agrees with Φ_i on G_i and satisfies (1) for $i+1$. $\Phi:=\bigcup_{i=1}^{\infty}\Phi_i$ is a strong embedding of G in G^2.

It is now easy to assemble a universal graph in $\mathcal{G}$ from copies of G^2. Put $G_1^*=G^2$. Having defined $G_1^*, \dots, G_i^*$ we obtain G_{i+1}^* by attaching disjoint copies of G^2 to G_i^* in the following way: for each vertex v of G_i^* and every $j\in\mathbb{N}$ we join infinitely many copies of G^2 to G_i^* identifying their j-th vertex with v. Then any given $G\in\mathcal{G}$ can be embedded in $G^*:=\bigcup_{i=1}^{\infty} G_i^*$ inductively along its block-cutvertex tree, so G^* is strongly universal in $\mathcal{G}$. ▮

Note that for finite $G\in\mathcal{G}^2$ the construction of Lemma 2.4 implies

$$|E(G_{i+1})\setminus E(G_i)| = \sum_{P\in\mathcal{P}_i} l(P) \leqq \sum_{P\in\mathcal{P}_i} 2\,|P\setminus G_i| = 2|G_{i+1}\setminus G_i|$$

for $i=1, 2, \dots$ with equality iff every $P\in\mathcal{P}_i$ has length 2. Therefore by $e(G_1)=1=2|G_1|-3$ we obtain as an immediate consequence of Lemma 2.4 a constructive proof of Dirac's result on the extremal size of graphs in $\mathcal{G}(TK^4)$ (see [1]):

2.6. Corollary. *For $n\geqq 2$ $\operatorname{ex}(n;\,TK^4)=2n-3$, and the extremal graphs are those that can be obtained from K^2 by repeated applications of the operation*

(2) *add a new vertex and join it to exactly two adjacent vertices.* ▮

(This construction of — in fact all — maximal graphs in $\mathscr{G}(TK^4)$ was first given by K. Wagner, see Section 5 below.)

Only little is known about the relationship between universal and strongly universal graphs. In [2] $\mathscr{G}(TK^{2,3})$ was shown to be an example of a class that contains a universal but no strongly universal graph. It would certainly be interesting to have a full characterization of all classes with this property.

One might also ask which graphs in general are strongly universal in some monotone class of graphs. In other words: which are the graphs that contain each of their subgraphs also as an induced subgraph?

3. The Case $5 \leq n < \aleph_0$

Let G be a graph, $\sigma > 0$ an ordinal, and G_λ induced[1] subgraphs of G for each $\lambda < \sigma$. The family $(G_\lambda)_{\lambda<\sigma}$ is said to form a (reduced) *simplicial decomposition* of G if

(i) $G = \bigcup_{\lambda<\sigma} G_\lambda$,

(ii) $(\bigcup_{\lambda<\tau} G_\lambda) \cap G_\tau =: S_\tau$ is complete for every τ, $0 < \tau < \sigma$,

(iii) no S_τ contains G_τ or any other G_λ, $0 \leqq \lambda < \tau < \sigma$.

The S_τ s are called *simplices of attachment*, but also any other complete graph may later be referred to as a simplex. A graph G is called *prime* if it cannot be decomposed in this way into more than one subgraph, which is easily shown to be equivalent to saying that G has no separating simplex. Every graph that contains no infinite simplex has a *prime decomposition*, that is, a simplicial decomposition each member of which is prime (this is a central theorem on simplicial decompositions; for a proof see [5]).

The members of any prime decomposition of such a graph G are always its maximally prime induced subgraphs, and its simplices of attachment are precisely those of its simplices that are relative cuts (an induced subgraph T of G is a *relative cut* if $V(T)$ separates two vertices x, y of G that are not separated by any proper subset of $V(T)$). Hence both are uniquely determined, i.e. independent of the decomposition chosen (for proofs see [4], [5]).

Let Γ be a non-empty set of finite graphs. Then every graph in $\mathscr{G} = \mathscr{G}(\Gamma)$ can be extended to a maximal element of $\mathscr{G}$ by adding edges (i.e. after adding any further edge the arising graph will no longer be in $\mathscr{G}$). Hence such classes $\mathscr{G}$ satisfy

(3.1) *If $\mathscr{G}$ has a universal element then $\mathscr{G}$ also has a universal element that is maximal in $\mathscr{G}$.*

The *homomorphism base* $\mathscr{B}(H\Gamma)$ of the class $\mathscr{G}(H\Gamma)$ (and the *subdivision base* $\mathscr{B}(T\Gamma)$ of $\mathscr{G}(T\Gamma)$) is the class of all graphs that occur as a member of a prime decomposition of some maximal element of $\mathscr{G}$. We remark that such base elements are always countable, even if we allowed $\mathscr{G}$ to contain uncountable as well as countable graphs; see [5, Ch. X] for details.

For the following two lemmas let $\mathscr{G} = \mathscr{G}(T\Gamma)$ or $\mathscr{G} = \mathscr{G}(H\Gamma)$, and let $\mathscr{B}$ be the base of $\mathscr{G}$.

[1] Technically, the requirement that each G_λ be induced in G could be dropped, since this follows anyhow by (i) and (ii).

3.2. Lemma. *Let $G \subset B \in \mathscr{B}$, and suppose that G is maximal in $\mathscr{G}$. Then $G=B$ or G is a simplex.*

Proof. By its maximality G is an induced subgraph of B. If C is a component of $B \setminus G$, let T be the set of vertices of G that are adjacent to a vertex of C. Then any two vertices x, y of T can be connected by an $x-y$ path through C that has only x and y in common with G. Hence the maximality of G implies that T spans a simplex S in B. But if $C \neq \emptyset$, then S separates B unless $S=G$, contradicting our assumption that B is prime. ∎

3.3. Lemma. *If $\mathscr{B}$ contains uncountably many pairwise non-isomorphic graphs that are maximal in $\mathscr{G}$, then $\mathscr{G}$ has no universal element,*

Proof. Suppose that $\mathscr{G}$ has a universal element G; by (3.1) we may assume G to be maximal. G contains a copy of every graph in $\mathscr{G}$, so in particular G contains uncountably many elements B of $\mathscr{B}$ that are maximal in $\mathscr{G}$ but no simplex. Now every such B is prime and by its maximality an induced subgraph of G. Since B cannot be a infinite simplex, this means that B is contained in some member B_τ of the prime decomposition of G (choose τ minimal such that $\bigcup_{\lambda \leq \tau} B_\lambda$ contains B; for a detailed proof see [4]). Hence by Lemma 3.2. the decomposition of G has uncountably many members, a contradiction. ∎

Lemma 3.3 can be used to derive a number of negative results concerning the existence of universal graphs. We start by constructing a certain class of infinite planar triangulations.

Let C_0, C_1, C_2, ... be a sequence of cycles embedded in the plane in such a way that the inner region of each C_i contains C_{i-1}. Let C_0 have length 1000, and let the other C_i's be of length 4 or 5. Add a vertex z (the center) inside C_0 and join it to all vertices of C_0. Furthermore, draw additional edges between each C_i and C_{i+1} in such a way that a planar triangulation arises that has no separating triangle, in which each vertex other than z has degree less than 300, and in which each vertex of C_i $(i \geq 1)$ is adjacent to some vertex of C_{i-1}. Since this graph depends on the 4—5 sequence α represented in the lengths of the C_i's, we denote it by T_α. Since z is the only vertex of degree 1000 and $V(C_i)$ is the i-th distance class from z, it is impossible that T_α is isomorphic to $T_{\alpha'}$ if $\alpha \neq \alpha'$. Therefore the T_α's form an uncountable class of non-isomorphic maximally planar prime graphs.

Now every maximally planar prime graph (which must be a simplex or 4-connected, see [5; p. 19—20]) is maximal in $\mathscr{G}(T\Gamma)$ and $\mathscr{G}(H\Gamma)$ if Γ is one of the sets $\{K^5\}$, $\{K^{3,3}\}$, $\{K^5, K^{3,3}\}$, $\{K^{1,2,3}\}$ or $\{K^{1,12,2}\}$. By Lemma 3.3 we have at once:

3.4. Theorem. *There is no universal element in $\mathscr{G}(T\Gamma)$ or $\mathscr{G}(H\Gamma)$ whenever Γ is one of* $\{K^5\}$, $\{K^{3,3}\}$, $\{K^5, K^{3,3}\}$, $\{K^{1,2,3}\}$ or $\{K^{1,1,2,2}\}$.

In particular, Pach's result [9] that there is no universal planar graph is reobtained.

If a graph B is prime and maximal in $\mathscr{G}(TG)$ (or $\mathscr{G}(HG)$) for some other graph G, then $B*1$ is prime and maximal in $\mathscr{G}(T(G*1))$ (or $\mathscr{G}(H(G*1))$), where $G*1$ denotes the graph arising from G by adding a new vertex and joining it to all vertices of G.

We therefore get by induction from 3.4 and 3.3:

3.5. Theorem. *$\mathcal{G}(TK^n)$ and $\mathcal{G}(HK^n)$ have no universal elements for any $n \geqq 5$.* ∎

More generally, it seems reasonable to conjecture that $\mathcal{G}(TG)$ (or $\mathcal{G}(HG)$) has a universal element whenever $\mathcal{G}(T(G*1))$ (or $\mathcal{G}(H(G*1))$) does.

4. Infinite Forbidden Subgraphs

De Bruijn (see [10]) proved that the class of all countable locally finite graphs has no universal element. It is therefore natural to consider the class of all countable graphs without an infinite path, since every infinite connected graph contains either a vertex of infinite degree or an infinite path.

The following theorem includes this case.

4.1. Theorem. *Let Γ be a non-empty class of countable graphs each containing an infinite path. Then $\mathcal{G} = \mathcal{G}(\Gamma)$ has no universal element.*

Proof. Let us first define for each countable ordinal λ a graph $G_\lambda \in \mathcal{G}$ by transfinite induction. Let G_μ be defined for all $\mu < \lambda$. To obtain G_λ, take the disjoint union of all G_μ, $\mu < \lambda$, add a vertex w_λ, and join it to all other vertices.

We have $w_\lambda \notin G_\mu \subset G_\lambda$ for all $\mu < \lambda$. Since w_λ is a cutvertex in G_λ, we easily see by transfinite induction that no G_λ contains an infinite path. As a countable union of countable graphs each G_λ is countable, and hence in $\mathcal{G}$.

Now suppose G^* is universal in $\mathcal{G}$. Then for all λ G^* contains a subgraph $G_\lambda^0 \simeq G_\lambda$.

Let us define vertices v_n, $n \geqq 1$, and subgraphs $G^* = G_0^* \supset G_1^* \supset G_2^* \supset \dots$ of G^* such that

(i) for each countable ordinal λ there is a $G_\lambda^n \subset G_n^*$ with $G_\lambda^n \simeq G_\lambda$, $n = 0, 1, \dots$,
(ii) $v_n \in V(G_{n-1}^*)$ for $n \geqq 1$,
(iii) $v_i \notin V(G_n^*)$ and $v_i v \in E(G^*)$ for all $0 < i \leqq n$ and $v \in V(G_n^*)$.

Put $G_0^* := G^*$, so for $n = 0$ (i) is true, and for (ii) and (iii) there is nothing to prove. Suppose we have v_i, $0 < i \leqq n$, G_n^* and G_λ^n as desired. Since there are uncountably many countable ordinals and $V(G_n^*)$ is countable, there is some vertex $v_{n+1} \in V(G_n^*)$ and an uncountable set U of countable ordinals such that v_{n+1} is the image of $w_\sigma \in G_\sigma$ in G_σ^n for all $\sigma \in U$. Put $G_{n+1}^* := \bigcup_{\sigma \in U} G_\sigma^n - v_{n+1}$.

Since U is uncountable, there is a $\sigma \in U$ with $\lambda < \sigma$ for each countable ordinal λ. Since $w_\sigma \notin G_\lambda \subset G_\sigma$, there is a $G_\lambda^{n+1} \simeq G_\lambda$ with $v_{n+1} \notin G_\lambda^{n+1} \subset G_\sigma^n$, i.e. $G_\lambda^{n+1} \subset G_{n+1}^*$. Hence (i) holds for $n+1$. By induction, (iii) for $n+1$ holds for all i, $0 < i \leqq n$, since $G_{n+1}^* \subset G_n^*$. By construction $v_{n+1} \notin V(G_{n+1}^*)$ and, furthermore, $v_{n+1} v \in E(G_n^*) \subset E(G^*)$ for all $v \in V(G_{n+1}^*)$, since for each $\sigma \in U$ w_σ is joined to all other vertices of G_σ. This completes the proof of (i)—(iii) for all $n \in \mathbb{N}$.

Now by (ii) and (iii) we have $v_i v_j \in E(G^*)$ for all $0 < i < j$. Hence the vertices v_n, $n \geqq 1$, span a $K^{\aleph_0}$ in G^*, contradicting $G^* \in \mathcal{G}$. ∎

4.2. Corollary. *$\mathcal{G}(TK^{\aleph_0})$ and $\mathcal{G}(HK^{\aleph_0})$ have no universal elements.* ∎

5. Related Results and Some More Open Problems

Let us go back to the line of proof followed in Section 3, and take a closer look at some of arguments involved.

It can be shown, for example, that the premise in Lemma 3.3 may be weakened to uncountability of the base itself. So whenever $\mathcal{G}(T\Gamma)$ or $\mathcal{G}(H\Gamma)$ has an uncountable base it has no universal element (see [3] for a proof).

Yet while it is easy to find cases in which a base contains graphs that are not themselves maximal in their class, no example is known of an uncountable base of which only countably many elements are maximal. In other words, it is not clear whether or not this strengthening of Lemma 3.3 means any real improvement.

It also seems difficult to decide, but would be very useful to know, to which extent the converse of Lemma 3.3 is true, for example which classes with a finite base have a universal element. The following example shows that this is not always the case.

5.1. Theorem. *There exists a set Γ for which $\mathcal{G}=\mathcal{G}(T\Gamma)$ has a finite base but no universal element.*

Proof. Let $\Gamma=\{C^4, K^{1,4}\}$, where C^4 denotes the cycle of length 4.

For any 2-connected $B\in\mathcal{B}(T\Gamma)$ let C be a cycle in B. By $C^4\in\Gamma$, C must be a triangle. Since B is prime, $B\setminus C$ is connected, and because B contains no TC^4, this means that at most one vertex x of C sends an edge to $B\setminus C$. So x is a cutvertex of B unless $B\setminus C=\emptyset$, that is $B=C\simeq K^3$. Hence $\mathcal{B}=\{K^1, K^2, K^3\}$.

For every 0—1 sequence $\alpha:\mathbf{N}\to\{0,1\}$ define a graph $G_\alpha\in\mathcal{G}$ as follows. Let G_α^0 be the graph of Fig. 3, and

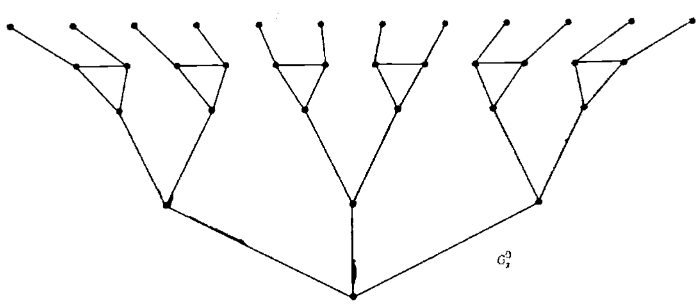

Fig. 3

G', G'' as shown in Fig. 4.

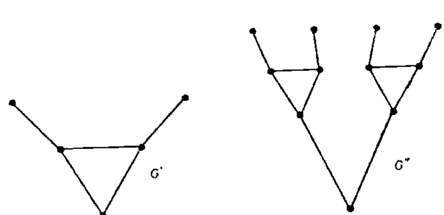

Fig. 4

Having defined $G_\alpha^0, \ldots, G_\alpha^{n-1}$, define G_α^0 from G_α^{n-1} by adding to it copies of G' (if $\alpha(n)=0$) or G'' (if $\alpha(n)=1$), one for each vertex y of degree 1 in G_α^{n-1}, identifying y with $x \in G'$ ($x \in G''$, resp.). Finally put $G_\alpha := \bigcup_{n \in \mathbf{N}} G_\alpha^n$.

For two 0—1 sequences $\alpha \neq \beta$ the graphs G_α and G_β are clearly non-isomorphic (note that only their vertices x_0 are of distance at least 2 from the nearest triangle and would have to be mapped onto one another). Since they are both 3-regular, their embeddings in any universal element G^* of $\mathcal{G}$ would therefore have to be disjoint, contradicting the fact that G^* would be countable. ▮

Note that in Theorem 5.1 we cannot replace $T\Gamma$ with $H\Gamma$, because each G_α contracts to a star of arbitrarily large order. But although it may not be entirely impossible that every class $\mathcal{G}(H\Gamma)$ with a countable base has a universal element, this does not seem very likely.

However, in most cases where the base $\mathcal{B}$ of a class $\mathcal{G}(T\Gamma)$ or $\mathcal{G}(H\Gamma)$ is known, the elements of $\mathcal{B}$ can be assembled to a universal graph in that class, similarly to the way in which we obtained the graph G^* in Theorem 2.5 from its blocks. By this method universal graphs can be constructed in the following cases.

5.2. Theorem. *Let $\mathcal{G}$ be any of the classes $\mathcal{G}(HK^5_{-1})$, $\mathcal{G}(HK^5_{-2})$, $\mathcal{G}(TC^5)$, $\mathcal{G}(HX^6)$ or $\mathcal{G}(TP_{\mathrm{III}})$, where K^5_{-1} is a K^5 minus an edge, K^5_{-2} a K^5 from which two independent edges have been deleted, C^5 the cycle of length 5, X^6 the wheel of order 6, and P_{III} the 'prism graph' $K^2 \times K^3$. Then $\mathcal{G}$ has a universal element.* ▮

It should be remarked that constructing the bases for the classes of Theorem 5.2 is non-trivial; for detailed references see [5, Ch. X, 7].

The main problem in the construction of a universal graph for a class $\mathcal{G}$ from the elements of its base $\mathcal{B}$ is that while $\mathcal{G}$ itself is normally uncountable, all its elements are to appear as subgraphs of a countable graph G^*, which means that often a member B_λ^* of the prime decomposition of G^* belongs to the decomposition of some $G \subset G^*$ that also contains $B_{\lambda+1}^*$, say, and at the same time to the decomposition of some other graph $G' \subset G^*$ that does not contain $B_{\lambda+1}^*$ but $B_{\lambda+2}^*$, say, which may not be in the decomposition of G. If in addition G and G' are such that $S_{\lambda+1}^*$ and $S_{\lambda+2}^*$ meet (e.g. if $S_{\lambda+1}^* = S_{\lambda+2}^* \subset B_\lambda^*$), then $B_{\lambda+1}^* \cup B_{\lambda+2}^*$ may well contain a forbidden configuration $X \in \Gamma$. Theorem 5.1 is a simple example of this.

An easy but crude way of curing this problem is to impose the following (rather strict) condition on $\mathcal{G}$, which is clearly sufficient for $\mathcal{G}$ to have a universal element if its base is countable.

(3) Let $G, G' \in \mathcal{G}$ such that $G \cap G'$ is a simplex of order at most n containing neither of G, G', where n is the maximum order of any simplex of attachment in a graph of $\mathcal{G}$. Then $G \cup G' \in \mathcal{G}$.

There are obvious ways of relaxing (3) a little. For example, G' may be required to be in $\mathcal{B}$, or, as a further restriction, we might only consider such G' that are actually a member of the prime decomposition of some $G'' \subset G \cup G'$ for which no embedding $G'' \to G$ exists that fixes $G'' \cap G$ (in which case the addition of G' to the universal graph under construction would be superfluous); this was done, for example, in the construction of the universal element of $\mathcal{G}(TK^{2,3})$ in [2]. Yet even then (3) seems far from being necessary for the existence of a universal element in a class with a countable base.

Let us finish by giving a positive example of the construction of a universal graph from the elements of the appropriate base using (3).

For $n \leqq 4$ let Γ_n denote the class of all n-connected finite graphs. For $n=3$ the base of $\mathcal{G}(H\Gamma_n)$ was determined by Wagner to be $\{K^1, K^2, K^3\}$ (see [5]). His characterization of $\mathcal{G}(H\Gamma_3)$ is equivalent to our Corollary 2.6, which is not surprising, since every 3-connected graph contains a TK^4 and therefore $\mathcal{G}(H\Gamma_3)=\mathcal{G}(TK^4)$.

For $n=4$ the base of $\mathcal{G}(H\Gamma_n)$ was in [6] shown to be $\mathcal{B}_4:=\{K^1, \dots, K^4, P_V, W\}$, where P_V is the 5-prism $C^5 \times K^2$ and W is a C^8 together with its four main diagonals. The largest order of any simplex of attachment S_λ in the prime decomposition $(B_\lambda)_{\lambda<\sigma}$ of an element G of $\mathcal{G}(H\Gamma_4)$ is 3, because every vertex in S_λ is joined to a vertex in $B_\lambda \setminus S_\lambda$, and B_λ does not contract to a K^5. Let G, G' be given as in (3), and suppose that $\Phi\colon V(G \cup G') \to V(X)$ is a contraction with $\varkappa(X) \geqq 4$. Since G and G' are in $\mathcal{G}$, X contains vertices v, v', whose inverse images under Φ are subsets of $V(G \setminus G')$ and $V(G' \setminus G)$, respectively. But then v and v' are separated in X by the at most 3 vertices of $\Phi(G \cap G')$, which contradicts $\varkappa(X) \geqq 4$. Therefore $\mathcal{G}(H\Gamma_4)$ satisfies (3) and contains a universal element.

This result complements a theorem proved in [3] which states that $\mathcal{G}(H\Gamma)$ has an uncountable base (and therefore no universal element) for $\Gamma=\{X \mid \alpha(X) \geqq n\}$, whenever $n \geqq 5$ and α denotes connectivity, edge connectivity or minimal degree. At first sight it is not at all clear whether all these classes satisfy (3) for $n \leqq 4$, or what their bases are. But every finite graph of minimal degree at least 4 subcontracts to a 4-connected graph (namely to a K^5 or a $K^{2,2,2}$ [7]), so these classes coincide for $n=4$ (as well as for $n \leqq 3$). We therefore have

5.3. Theorem. *Let $\Gamma=\{X \mid \alpha(X) \geqq n\}$, where α denotes connectivity, edge connectivity or minimal degree. Then $\mathcal{G}(H\Gamma)$ has a universal element if and only if $n \leqq 4$.* ∎

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